Handout[1] 3

Sept 13, 2022

[Topics]:
Counting
Permutation
Combination

1 Counting

Theorem 1.1 (Counting Rule) For k sets, each with n_i ($i = \{1, 2, 3, \cdots k\}$) possible putcomes. If pick one outcome from each set, then there will be $n_1 n_2 \cdots n_k$ possible ways to do so.

Example 1.1 Roll a die twice, each roll will have outcomes in $\{1, 2, 3, \cdots 6\}$. The possible outcomes combinations are: {1,1},{1,2},{1,3},$\cdots$,{1,6},{2,1},{2,2},$\cdots$,{2,6},$\cdots$,{6,1},{6,2},$\cdots$,{6,6}. There are $6 \times 6 = 36$ possible ways in total.

Pick with Replacement A set with n elements, if pick k times, each time pick 1 element with replacement, i,e. pick 1 element, then put it back. Then there are n^k ways to do so.
By the counting rule, each time there will be n possible ways to pick. So it follows $\underbrace{n \times n \times n \times \cdots \times n}_{k} = n^k$.

Pick without Replacement A set with n elements, if pick k times, each time pick 1 element without replacement, i,e. pick 1 element and then keep it without putting it back. Then there will be $n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$ possible ways to do so.
For the i_{th} pick, there will be $n - i + 1$ choices. So plug $n_1 = n, n_2 = n - 2, \cdots, n_k = n - k + 1$ into the counting rule.

Note that for the above "pick" problems, the order matters. For example, if there is a bag with black balls and red balls, and we pick form the bag twice. Then the event "first pick a black, then pick a red ball" and the event "first pick a red ball, then pick a black ball" are different though the final outcomes are all "1 balck and 1 red". The above "pick" problems are talking about the possible ways to do the pick experiment, so the order matters. "pick black then red" and "pick red then black" are two different ways to achieve the goal "pick 1 balck and 1 red."

There are cases when the order doesn't matter. For example, for the bag with black balls and red balls. If pick two balls from it, the possible outcomes will be "1 balck 1 red", or "2 black balls", or "2 red balls". For the event "1 black and 1 red", it doesn't matter whether you take the black ball out of the bag first or take the red one out first.

Example 1.2 There is a bag with 3 red balls and 2 blue balls. If pick one ball from the bag each time and pick twice. How many possible ways to do so? What is the possible way to pick "1 red and 1 blue"?
Solution: Let's denote the balls as: red1, red2, red3, and blue1, blue2. We pick twice, each with 1 ball. By the counting rule, there are 5×5=25 ways to do the pick. To pick "1 red and 1 blue", the posible ways are shown in **Table 1**. There are in total 12 ways to do the pick. Hence the probability of "the outcome will be 1 red and 1 blue" is $\frac{12}{25}$.

Example 1.3 There is a bag with 3 red balls and 2 blue balls. If pick two balls from the bag, how many possible ways to do so? What is the possible way to pick "1 red and 1 blue"?
Solution: Let's denote the balls as: red1, red2, red3, and blue1, blue2. We pick two balls at one time. **Table 2** shows all the possible ways to do the pick. There are 6 ways to pick "1 red 1 blue".

In the next two sections, we take a more general view on this.

[1]This handout is made by Hongkai Wang for Econ 329 Economic Statistics. It is adapted from Professor Wiseman's lectures, however, all errors are mine.

#	First pick	Second pick
1	red1	blue1
2	red1	blue2
3	red2	blue1
4	red2	blue2
5	red3	blue1
6	red3	blue2
7	blue1	red1
8	blue1	red2
9	blue1	red3
10	blue2	red1
11	blue2	red2
12	blue2	red3

Table 1: **Example 1.2** Ways to choose 1 red and 1 blue

#	outcome
1	red1,red2
2	red1,red3
3	red1,blue1
4	red1,blue2
5	red2,red3
6	red2,blue1
7	red2,blue2
8	red3,blue1
9	red3,blue2
10	blue1,blue2

Table 2: **Example 1.3** Ways to pick two balls

2 Permutation (order matters)

Theorem 2.1 For n elements, there are $n! = 1 \times 2 \times 3 \times \cdots \times n$ ways to order them.

To make the n elements into an ordered sequence, we do the following way:
Step 1 : Pick 1 out of the n elements, put it in the first place. There are $n-1$ elements remain unordered.
Step 2 : Pick 1 out of the $n-1$ elements, put it in the second place. There are $n-2$ elements remain unordered.
...
Step n-1 : Pick 1 out of the 2 elements, put it in the $(n-1)_{th}$ place. There is 1 element remains unordered.
Step n : Pick 1 out of the 1 element, put it in the last place. All elements are ordered.

By the counting rule, the total number of possible ways to make the order is $n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$.

Theorem 2.2 For n elements, the permutation(the order matters) of take k of them is:

$$\mathbf{P}_{n,k} = \frac{n!}{(n-k)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1).$$

The method is the same as above:
Step 1 : Pick 1 out of the n elements, put it in the first place. There are $n-1$ elements remain unordered.
Step 2 : Pick 1 out of the $n-1$ elements, put it in the second place. There are $n-2$ elements remain unordered.
...
Step k-1 : Pick 1 out of the $n-k+2$ elements, put it in the $(k-1)_{th}$ place. There are $n-k+1$ elements remains unordered.
Step k : Pick 1 out of the $n-k+1$ elements, put it in the k_{th} place.

By the counting rule, the total number of possible ways to make the order is $n \times (n-1) \times (n-2) \times \cdots \times (n-k+2) \times (n-k+1)$.

Example 2.1 Professor Wiseman is going to (randomly) choose 10 students from the Econ 329 class with total 200 students to do a presentation. How many ways to arrange it?
Solution: $\mathbf{P}_{200,10} = \frac{200!}{(200-10)!} = 81470204436547390464000$. Note that the order matters since "Mike do the presentation first, then Lily do the presentation" and "Lily do the presentation first, then Mike" are different arrangements.

3 Combination (order doesn't matter)

Theorem 3.1 There are n elements. The number of possible ways to choose k(regardless of order) is:

$$\mathbf{C}_{n,k} = \frac{\mathbf{P}_{n,k}}{k!} = \frac{n!}{(n-k)!k!}.$$

Recall the permutation $\mathbf{P}_{n,k}$ is the number of ways to choose k out of n elements and make them ordered. Recall Theroem 2.1, for k elements, there are $k!$ ways to order them. So the combination $\mathbf{C}_{n,k}$ simply follows by the total number of ordered ways divided by how many possible ways to order.

Notation The combination $\mathbf{C}_{n,k}$ is also denoted by: $\binom{n}{k}$, meaning choose k from n.

Example 3.1 Professor Wiseman is going to choose 10 students from the Econ 329 class of 200 students to fight with Alabama. How many possible ways to make the team?
Solution: $\binom{200}{10} = \mathbf{C}_{200,10} = \dfrac{200!}{(200-10)!10!} = 22451004309013280.$

Example 3.2 Roll 3 dice, what is the probability of each number is different?
Solution: There are $6 \times 6 \times 6 = 216$ possible outcomes. For the first die, it can be any number in $\{1,2,3,4,5,6\}$. So there are $\binom{6}{1} = \frac{6!}{(6-1)!1!} = 6$ ways to choose. For the second die, it should be different form the first die, so it can be choosen from the rest 5 numbers. Then there are $\binom{5}{1} = \frac{5!}{(5-1)!1!} = 5$ ways to choose. Similarly, there 4 ways to choose for the third die. So the probability will be:

$$\frac{\binom{6}{1} \times \binom{5}{1} \times \binom{4}{1}}{6 \times 6 \times 6} = \frac{5}{9}.$$

Example 3.3 10 students are walking on the speedway. Each student has a probability of 0.8 to receive a flyer. What is the probability 3 of them receive flyers? What is the probability at least one of them receive a flyer?
Solution: $\binom{10}{3} \times 0.8^3 \times (1-0.8)^7 = \frac{10!}{(10-3)!3!} \times 0.8^3 \times 0.2^7 = 0.000786432.$
$\mathbb{P}$(at least 1 receive flyer)=1-$\mathbb{P}$(0 of them receive flyer)=1-$\binom{10}{0} \times 0.8^0 \times 0.2^{10} \approx 1.$

Good Luck on Your Midterm!