Handout[1] 8

Nov 7, 2022

[Topics]:

Limiting Theorems

1 Some Inequalities

Markov Inequality
Suppose a random variable X has finite $k'th$ moment, i.e. $E(|X|^k) < \infty$. Then for any $\epsilon > 0$,

$$P(|X| \geq \epsilon) \leq \frac{E(|X|^k)}{\epsilon^k}$$

Proof
Assume X is a continuous random variable with pdf $f(x)$, then

$$\begin{aligned}
P(|X| \geq \epsilon) &= \int_{|X| \geq \epsilon} f(x)dx \\
&\leq \int_{|X| \geq \epsilon} (\frac{|x|}{\epsilon})^k f(x)dx, \quad as \ |x| \geq \epsilon \\
&= \frac{1}{\epsilon^k} \int_{|X| \geq \epsilon} |x|^k f(x)dx \\
&\leq \frac{1}{\epsilon^k} \int_{-\infty}^{+\infty} |x|^k f(x)dx \\
&= \frac{1}{\epsilon^k} \times E(|X|^k) \\
&= \frac{E(|X|^k)}{\epsilon^k}
\end{aligned}$$

The proof for discrete random variable case is similar.

Corollary of Markov Inequality
Suppose a random variable X satisfies the condition that $E(X^2) < \infty$, then for any $a > 0$, we have:

$$P(|X| \geq a) \leq \frac{E(X^2)}{a^2}$$

Proof For Markov inequality, take $k = 2$.

Chebyshev Inequality
Suppose a random variable X satisfies the condition that $E(X^2) < \infty$, then for any $\epsilon > 0$, we have:

$$P(|X - E(X)| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2}$$

Proof Let $Y = X - E(X)$ and apply the above inequality:

$$\begin{aligned}
P(|X - E(X)| \geq \epsilon) &\leq \frac{E(X - E(X))^2}{\epsilon^2} \\
&= \frac{Var(X)}{\epsilon^2}
\end{aligned}$$

[1] This handout is made by Hongkai Wang for Econ 329 Economic Statistics. It is adapted from Professor Wiseman's lectures, however, all errors are mine.

2 Weak Law of Large Numbers

Suppose $X_i, X_2, \cdots, X_n$ are independent random variables with common mean μ and common finite variacne σ^2. Let $\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$ denotes the sample mean. Then for any $\epsilon > 0$, we have

$$\lim_{n\to\infty} P(|\overline{X_n} - \mu| > \epsilon) = 0$$

Proof

$$\begin{aligned}
E(\overline{X_n}) &= E(\frac{1}{n}\sum_{i=1}^{n} X_i) \\
&= \frac{1}{n}\sum_{i=1}^{n} E(X_i) \\
&= \mu
\end{aligned}$$

And

$$\begin{aligned}
Var(\overline{X_n}) &= Var(\frac{1}{n}\sum_{i=1}^{n} X_i) \\
&= (\frac{1}{n})^2\sum_{i=1}^{n} E(X_i) \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

Now apply Chebyshev's inequality,

$$\begin{aligned}
P(|\overline{X_n} - E(\overline{X_n})| \geq \epsilon) \leq \frac{Var(\overline{X_n})}{\epsilon^2} \\
P(|\overline{X_n} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}
\end{aligned}$$

As $n \to \infty$, we have $\lim_{n\to\infty} P(|\overline{X_n} - \mu| > \epsilon) = 0$ for any $\epsilon > 0$.

Interpretation
For any observations of a distribution, as the size of the observation sample goes larger, the sample mean will converge to the mean of the distribution.

3 Central Limit Theorem

Suppose $X_i, X_2, \cdots, X_n$ are independent random variables with common mean μ and common finite variacne σ^2. Let $\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$ denotes the sample mean. Then for each fixed x, we have

$$\lim_{n\to\infty} P(\frac{\sqrt{n}(\overline{X_n}-\mu)}{\sigma} \leq x) = \Phi(x)$$

where $\Phi(x)$ denotes the cdf of standard normal distribution.

Interpretation
For any observations of a distribution, as the size of the observation sample goes larger, the sample mean will behave as a normal distribution.
As in section 2, $E(\overline{X_n}) = \mu$ and $Var(\overline{X_n} = \frac{\sigma^2}{n})$. Then

$$\begin{aligned}
E(\frac{\sqrt{n}(\overline{X_n} - \mu)}{\sigma}) &= \frac{\sqrt{n}}{\sigma}E(\overline{X_n} - \mu) \\
&= \frac{\sqrt{n}}{\sigma} \times 0 \\
&= 0
\end{aligned}$$

And

$$\begin{aligned} Var(\frac{\sqrt{n}(\overline{X_n} - \mu)}{\sigma}) &= (\frac{\sqrt{n}}{\sigma})^2 Var(\overline{X_n} - \mu) \\ &= \frac{n}{\sigma^2} \times \frac{\sigma^2}{n} \\ &= 1 \end{aligned}$$